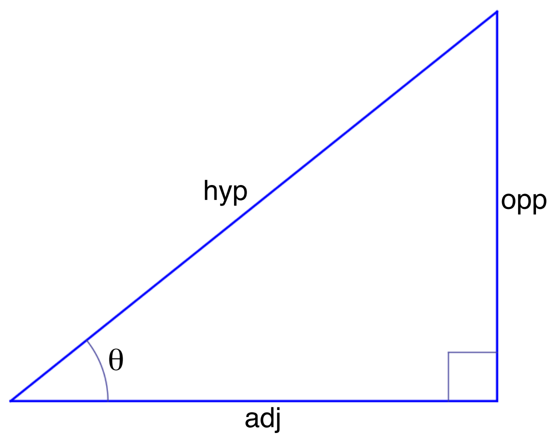
Trig: a refresher

---

1 Basic triangle definition

When it comes to defining trig functions, we always start with a right triangle, pictured below. The trig functions are then ratios between aspects of this triangle.



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \csc(\theta) = \frac{\text{hyp}}{\text{opp}}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sec(\theta) = \frac{\text{hyp}}{\text{adj}}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} \qquad \cot(\theta) = \frac{\text{adj}}{\text{opp}}$$

CAUTION: $\csc\theta$ is NOT $1/\cos\theta$; it's $1/\sin\theta$. The "co" doesn't match. This is something people often mix up.

We also can see, immediately, two important trig identities:

$$\boxed{\sin^2\theta + \cos^2\theta = 1 \text{ (thanks, Pythagoras), and } \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta.}$$

A little harder to see, but we also get some interesting *double-angle formulas* for sin and cos:

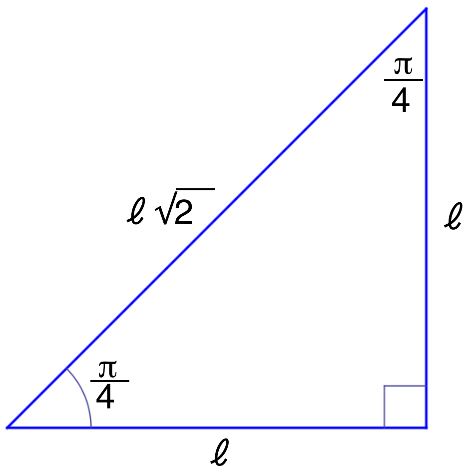
$$\boxed{\sin(2\theta) = 2\sin\theta\ \cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta}$$

Note: not something you need to memorize. But you should know the existence of.

In general, calculating these ratios given an angle is not easy. However, there are two triangles that are very important to remember:

The half-square

This is exactly what it sounds like: take a square and cut it along the diagonal to make a triangle. This gives us a right angle $\frac{\pi}{2}$, while the other two angles are both $\frac{\pi}{4}$. Then, the side lengths are as shown on the picture (thanks, Pythagoras).

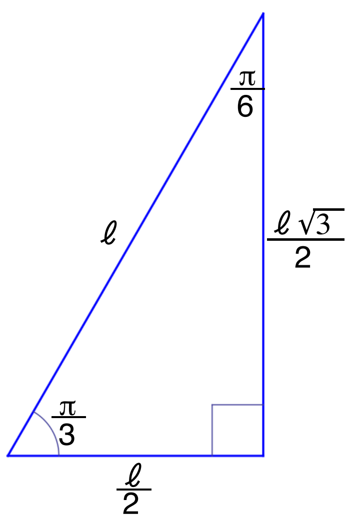


Then we can read off the following values:

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\tan\left(\frac{\pi}{4}\right) = 1$$

The half-equilateral

This time around, we cut an equilateral triangle in half to make the right triangle below. Again, we can thank our friend Pythagoras for the side lengths.



Then we can read off the following values:

$$\cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$
$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

We summarize these values in the following table:

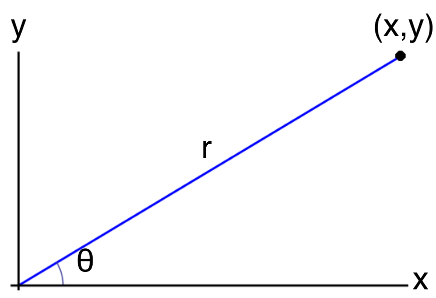
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	N/A

Note: you should be able to figure out values of $\csc\theta$, $\sec\theta$, and $\cot\theta$ for these angles from this table.

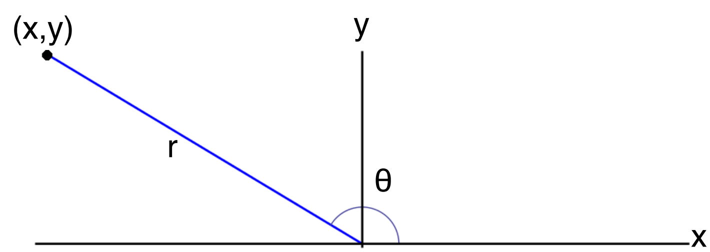
2 The unit circle

The triangle definition of our trig functions is great, but we can see its limitations - what about when the angle isn't acute? We still want to be able to understand $\sin\theta$, $\cos\theta$ and $\tan\theta$ for larger values of θ.

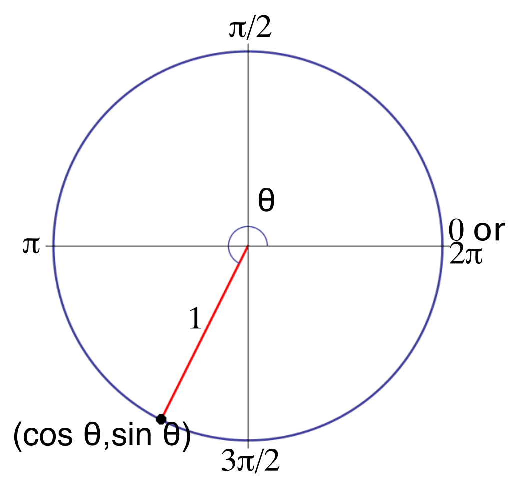
If we put our triangle, with hypotenuse r, in the first quadrant, we can realize $r\sin\theta$ and $r\cos\theta$ as the x and y coordinates of the top vertex, as pictured.



We use this idea to expand our understanding of our trig functions:



We usually take $r = 1$, then $\sin\theta$ and $\cos\theta$ are exactly the x and y coordinates of the point of intersection of the unit circle and the line that makes an angle of θ with the positive real axis. Then $\tan\theta$ is the slope of that line.



With this unit circle picture, we can easily remember:

$$\sin(\pi) = 0, \quad \cos(\pi) = -1, \quad \sin\left(\frac{3\pi}{2}\right) = -1, \quad \cos\left(\frac{3\pi}{2}\right) = 0,$$

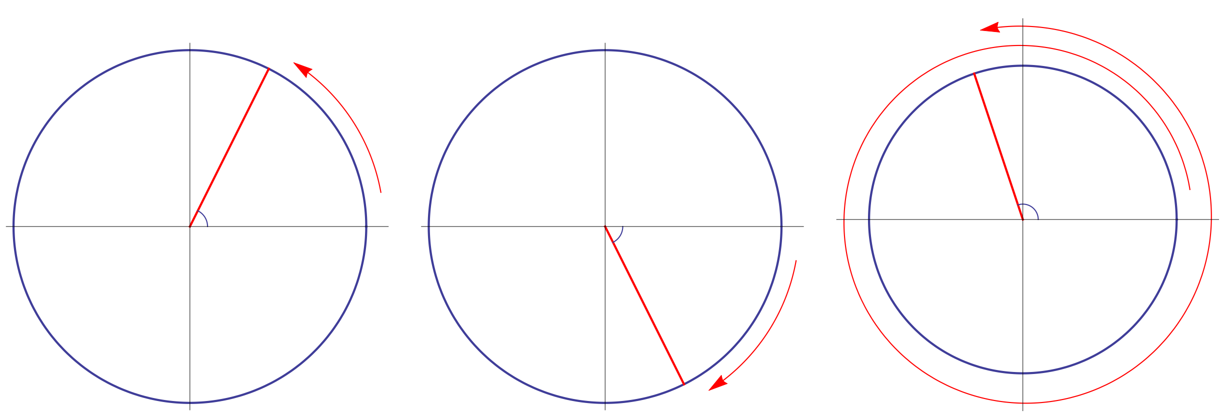
and we can extend the table from the last section to determine more values of our trig functions, for example:

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$\sin\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\cos\theta$	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\tan\theta$	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	-1

We also see a few more interesting identities:

$$\cos(\pi + \theta) = -\cos\theta, \qquad \sin(\pi + \theta) = -\sin\theta.$$

But why limit ourselves to θ between 0 and 2π? Angles can be thought of as "directions for turning." So we can talk about θ between 0 and 2π (turn counterclockwise), but also negative values of θ (turn clockwise), or values larger than 2π or smaller than -2π (spin in circles).



$\theta > 0$ $\qquad$ $\theta < 0$ $\qquad$ $\theta > 2\pi$

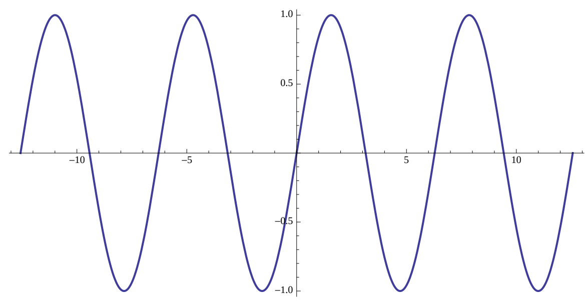
From this formulation, we also see the *periodicity* of sin, cos, and tan:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta \qquad \tan(\theta + 2\pi) = \tan\theta,$$

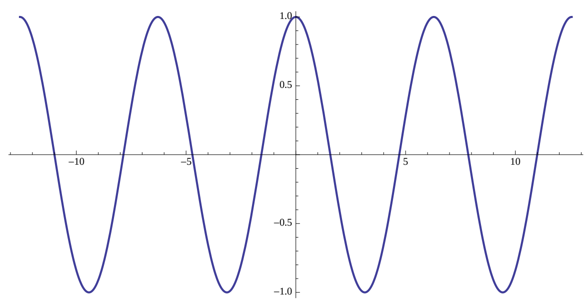
and some *symmetry* in these functions:

$$\boxed{\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta \qquad \tan(-\theta) = -\tan\theta.}$$

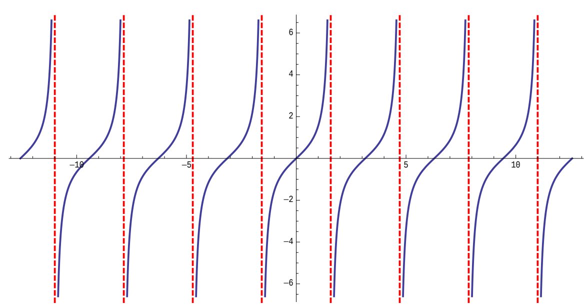
We also get the graphs:



$\sin(x)$



$\cos(x)$



$\tan(x)$

Note: the graph of $\tan\theta$ has *vertical asymptotes* whenever $\theta = \frac{\pi}{2} + N$, for any integer N. This is because those are exactly the values where $\cos\theta = 0$.

3 Inverse trig

A *function* f is a machine. It takes an *input*, x, does something to it, and gives us *a unique output*, $f(x)$. And it has to be the same every time: if you put 5 in to the machine once and it gives you 17, it must give you 17 *every other time* you give it 5. But, in general, the machine can send multiple, different inputs to the same output: 5 gives 17, but 1000 could also give 17. That's still allowed for the machine to be a function.

So, we need a special name for when *each output* can only be obtained from *a unique input*. In these cases, we call the function *one-to-one* or *1-to-1*: there is *one* output for each input *AND* there is *one* input for each output.

Examples of 1-to-1 functions: $y = mx + b$ for any m and b, $y = x^3$, $y = e^x$.

Examples of functions that are not 1-to-1: $y = 5$, $y = x^2$.

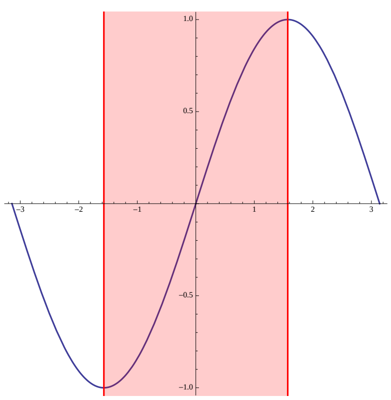
Why do we like 1-to-1 functions? Because we can *undo* them. We can answer the question, "Given $y = f(x)$, find x." We can make another machine to do this for us and we call it the *inverse*. It gets the fancy name: f^{-1}.

CAUTION: this is NOT $1/f(x)$!!! That is not what this notation means!!

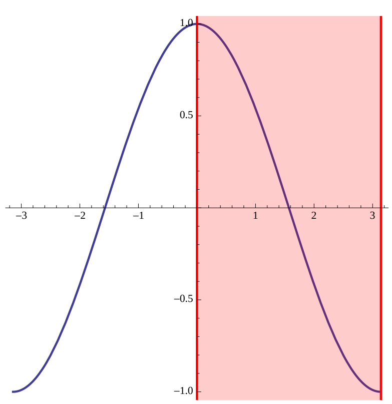
If f^{-1} is the inverse of f, it means that

$$\boxed{f(x) = y \text{ and } f^{-1}(y) = x.}$$

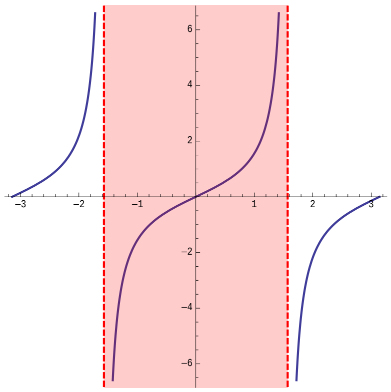
Important point: a function may not be 1-to-1 but often, we can adjust the *domain* to get a 1-to-1 function. Consider our trig functions: as we saw in the last section, they are definitely *not* 1-to-1 over their entire domains. But **we can restrict the domain to get 1-to-1 functions**:



restricted domain for $\sin(x)$ is $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



restricted domain for $\cos(x)$ is $0 \leq x \leq \pi$.



restricted domain for $\tan(x)$ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

These domains are what we use to define *inverse trig functions*. (There are two notations for these, so both are included.)

$$\begin{array}{|c|}\hline \sin^{-1}(x) = \arcsin(x) = \theta \text{ means that } \sin\theta = x \\ \cos^{-1}(x) = \arccos(x) = \theta \text{ means that } \cos\theta = x \\ \tan^{-1}(x) = \arctan(x) = \theta \text{ means that } \tan\theta = x \\ \hline \end{array}$$

It is important to note:

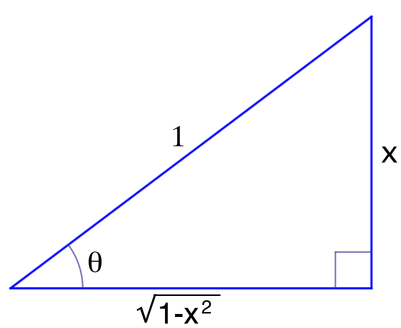
- The domain of arcsin is $-1 \leq x \leq 1$; the range is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- The domain of arccos is $-1 \leq x \leq 1$; the range is $0 \leq \theta \leq \pi$.
- The domain of arctan is ALL real numbers; the range is $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

REMEMBER: $\sin^{-1}(x) = \arcsin(x)$ is NOT the same as $\csc(x) = \frac{1}{\sin x}$.

Fun fact 1: just as the graph of $\tan\theta$ had vertical asymptotes at $\theta = \frac{\pi}{2} + N$, the graph of $\arctan x$ has *horizontal* asymptotes at $y = \frac{-\pi}{2}$ and $y = \frac{\pi}{2}$. This means

$$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2} \qquad \text{and} \qquad \lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}.$$

Fun fact 2: from the triangle, we can actually evaluate things like $\cos(\arcsin x)$. Suppose $\arcsin x = \theta$. So $\sin\theta = x$ and we have the triangle at right.



Then, we can read off that

$$\cos(\theta) = \cos(\arcsin x) = \frac{\text{adj}}{\text{hyp}} = \sqrt{1 - x^2}.$$