Intervals and Sets

1. Sets

In mathematics, a **set** is a collection of elements. It can be defined by listing all its elements, or by a mathematical property.

If S is a set, and x is an element of this set, then we write $x \in S$ to mean "x belongs to S". To mean that x is not an element of S, we write $x \notin S$.

To define a set, we can write its elements between curly brackets $\{\}$, for example:

- The set $S = \{1, 2, 3\}$ is the collection formed by the numbers $1, 2$ and 3. So we have $1 \in S$ but $6 \notin S$.
- The set of all of my pets is $P = \{\text{dog}, \text{parrot}, \text{rabbit}, \text{elephant}\}$. So $\text{rabbit} \in S$ but $\text{cat} \notin P$.
- Sets don't have to contain only numbers or only letters. We can combine any type of mathematical symbols: $\{6, \text{Eiffel Tower}, \circ, +, \text{flower}\}$ is also a set.

Sometimes we can't write down all the elements of the set, so we need a different way. For example, we can write $\mathbb{R}$ for the set of all real numbers, or $\mathbb{Z}$ for the set of relative integers. An interval is also a set.

Sets can also be defined by one or several mathematical properties, for example "all relative integers that are divisible by 2 and have at least 3 prime factors", or "all positive real numbers", or, even simpler "all rational numbers". To write this in a concise manner, we use more mathematical symbols.

Let's start with an example: the set of all relative integers that are greater than or equal to -2 would be:

$$\{x | x \in \mathbb{Z} \text{ and } x \geq -2\}.$$

- The curly brackets $\{\}$, as above, mean "the set of...". Inside them, we write the list of the elements of the set (as an enumeration, or with a mathematical property).
- Then comes the names of the elements in the set: here we'll call them x.
- The vertical bar $|$ (sometimes appears as a column or a comma), can be read "such as...". It separates the name of the variable and the defining properties.

- On the right hand side, the defining properties of the set, separated by commas or by logical connectors ("and", "or", etc): here we want all the elements that are relative integers AND that are greater than or equal to -2.

Note that the rules aren't completely fixed, and we can add a property on the left side of the vertical bar. Thus

$$\{x \in \mathbb{Z} | x \geq -2\}$$

is also valid, and denotes the same set as above.

Another example is the definition of the graph of a function that we saw in class. Recall that for a function f with domain D, the graph of f is defined as:

$$\{(x, f(x)) | x \in D\}.$$

- Again, the curly brackets tell us we are defining a set.
- The elements of this set are pairs (x, y) where $y = f(x)$.
- Additionally, we require that x belongs to D, the domain of f (otherwise $f(x)$ wouldn't exist).

This reads as "the set of the pairs $x, f(x)$ such that x belongs to D". We could also have written it:

$$\{(x, y) | x \in D \text{ and } y = f(x)\}.$$

As you see, there's a bit of flexibility in how things are written.

Remark. If you don't know how to express a property in mathematical notation, you can always spell it out. For example, the set of all odd integers can be written $\{n \in \mathbb{Z} | n \text{ is odd}\}$.

Exercises.

1. Write the following sets in mathematical notation:
   - All real numbers except 3.
   - All rational numbers with a coprime numerator and denominator.
   - The graph of the function $x^2 + 6x + \frac{1}{x}$.
2. Express the following sets in words and name two elements that belong to them.
   - $\{y \in \mathbb{Z} | y < 8\}$
   - $\{w \in \mathbb{R} | w^2 \neq 2\}$

- $\{n|n \in \mathbb{N} \text{ and } 2^n > 1000\}$

TL;DR: Write sets between curly brackets. In it, write the name of the variable, and maybe what type of variable it is (integer, real number...). Then a vertical bar, and the defining properties of the elements of the set.

2. Intervals

Intervals are a special kind of sets: they're sets of real numbers. When we have two real numbers a and b with $a < b$, the **interval** $[a, b]$ is the set of all numbers between a and b. There are two types of brackets we can use for intervals:

- The square brackets "$[$" and "$]$". If we write an interval $[a, b]$, then we mean "all numbers between a and b, *including a and b*". So, for example, in the interval $[0, 1]$, we have $0.7,\ 0.1$ but also 1 and 0.
- The round brackets $()$. If we write (a, b) then we mean "all numbers between a and b, but not including a or b."

We can combine the brackets. For example, we can write $[-2, 1000)$ to mean "all numbers between -2 and 100, including -2 but excluding 1000", or $(-2.7, 6]$ to mean "all numbers between -2.7 and 6, not including -2.7 but including 6.

If we don't want to "close" the interval, for example if we want to write "all numbers greater than 2", we can use an infinity symbol: ∞. Recall that the bracket is always round on the side of the infinity symbol *because ∞ is not a number*.

TL;DR:

- $x \in [a, b]$ if and only if $a \leq x \leq b$.
- $x \in (a, b)$ if and only if $a < x < b$.
- $x \in (a, b]$ if and only if $a < x \leq b$.
- $x \in [a, b)$ if and only if $a \leq x < b$.
- $x \in [a, \infty)$ if and only if $x \geq a$.
- $x \in (a, \infty)$ if and only if $x > a$.
- $x \in (-\infty, b]$ if and only if $x \leq b$.
- $x \in (-\infty, b)$ if and only if $x < b$.

3. Union, intersection, difference

If A and B are sets, then we write:

- $A \cup B$ for the set of all elements that are in A or in B (they can be in both), and we say "A union B".
- $A \cap B$ for the set of all elements that are in A and in B. We say "A intersection B" (or A intersected with B).
- $A \backslash B$ for the set of all elements that are in A but not in B.

Exercises.

1. Write the following sets as intervals or unions of intervals:
   - The set of all real numbers.
   - The set of all nonnegative numbers, except 3.
   - The set of all real numbers that are less than -3, or more than 6.
2. Write the following intersections as one interval, and give two elements in it:
   - $(-\infty, 1] \cap (0, \sqrt{2})$.
   - $[5, 800) \cap (6, 10)$.
   - $(-\pi, \infty) \cap [-\pi, 16]$